On the linear automorphism group on a $*$-algebra (Categories, and Combinatorial Representation Theory Series) CCRT[7]

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Outline

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Goals

The semi-simple algebra $\mathcal{K}(n)$ of ribbon graphs is constructed from $\mathcal{A} = \mathbb{C}(S_n)^{\otimes 2}$, by "quotienting" it by the S_n-diagonal adjoint action on the tensor product;

• There are vectors T_r spanning the center of $\mathcal{K}(n)$ that have integral matrices.

• T_r are useful to identify the dimensions of the WA - matrix decomposition of $\mathcal{K}(n)$

• These dimensions are nothing but the square of Kronecker coeff.: they can be computed by a triangularization algorithm applied on the stack of matrices of the T_r's [Ramgoolam & BG [2010.04054]].

We may ask:
What is the most generic setting on semi-simple algebras for which this result generalizes?

(finding a 'nice' basis of the centre of an 'invariant' semi-simple sub-algebra of a given algebra)

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$\mathcal{K}(n)$, the graph algebra

- Group algebra $\mathbb{C}(S_n)$, i.e. an element of which writes $a = \sum_{\sigma \in S_n} \lambda_\sigma \sigma, \ \ \lambda_\sigma \in \mathbb{C}$
- Back to coset formulation: Consider the orbits

$$(\sigma_1, \sigma_2) \sim (\gamma\sigma_1\gamma^{-1}, \gamma\sigma_2\gamma^{-1}) \quad (1)$$

- Define $\mathcal{K}(n) \subset \mathbb{C}(S_n)^{\otimes 2}$ is the vector space over $\mathbb{C}$

$$\mathcal{K}(n) = \mathrm{Span}_{\mathbb{C}}\Big\{ \sum_{\gamma \in S_n} \gamma\sigma_1\gamma^{-1} \otimes \gamma\sigma_2\gamma^{-1}, \ \sigma_1, \sigma_2, \in S_n \Big\} \quad (2)$$

$\to$ Fact : an orbit $\mathrm{Orb}(r)$ is 1-1 correspondence with a base vector E_r of $\mathcal{K}(n)$.

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$\mathcal{K}(n)$, the graph algebra

- Take a base element of $\mathcal{K}(n)$

$$A_{\sigma_1,\sigma_2} = \sum_{\gamma \in S_n} \gamma\sigma_1\gamma^{-1} \otimes \gamma\sigma_2\gamma^{-1} \tag{3}$$

- Associative multiplication

$$A_{\sigma_1,\sigma_2} A_{\sigma_3,\sigma_4} = coeff. \sum_{\tau \in S_n} A_{\sigma_1\tau\sigma_3\tau^{-1},\, \sigma_2\tau\sigma_4\tau^{-1}} \tag{4}$$

- There is a pairing

$$\boldsymbol{\delta}_2(\otimes_{i=1}^2 \sigma_i ; \otimes_{i=1}^2 \sigma_i') = \prod_{i=1}^{2} \delta(\sigma_i \sigma_i'^{-1}) \tag{5}$$

that extends by linearity to $\mathcal{K}(n)$ and that is non-degenerate.

Theorem (BG, Ramgoolam '17)

$\mathcal{K}(n)$ is an associative semi-simple algebra with unit element.

Outline

$*$-Algebra and states

Consider $\mathcal{A}$ an associative algebra with unit over the complex field $\mathbb{C}$, its neutral element will be noted e.

- We call *involution* within $\mathcal{A}$, a bijection $x \to x^*$ which is additive, semi-linear and an involutive anti-automorphism. The pair $(\mathcal{A}, *)$ is called *star-Algebra*.

- $\mathcal{C}_+(\mathcal{A})$ is the set of elements of the form $\sum_{i \in F} x_i x_i^*$ (where F finite). It is a real convex cone within $SA(\mathcal{A})$ (set of self-adjoint elements, i.e. such that $x = x^*$).

- $State(\mathcal{A})$ is the set of linear forms $f \in \mathcal{A}^*$, the dual of $\mathcal{A}$, such that

$$z \in \mathcal{C}_+(\mathcal{A}) \Longrightarrow f(z) \geq 0 \quad \text{and} \quad f(1) = 1, \tag{6}$$

where 1 is the constant function on $\mathcal{A}$.

- A semi-positive non degenerate state (SPS) $f \in State(\mathcal{A})$ satisfies

$$z \in \mathbb{C}_+(\mathcal{A}) \text{ and } f(z) = 0 \Longrightarrow z = 0. \tag{7}$$

We also call a SPS, a faithful state.

$\mathcal{A}$ an $*$-algebra and Hilbert space

Notable facts:

1. We start with a finite dimensional $*$-algebra $\mathcal{A}$ and remark that e^* is neutral so that $e^* = e$.

2. Now, $\mathcal{A}$ is equipped with a SPS φ as in (7). With φ, we build the following 2-form

$$g(x, y) = \langle x|y\rangle = \varphi(x^* y) \tag{8}$$

which satisfies

$$\langle ax|y\rangle = \langle x|a^* y\rangle \tag{9}$$

3. One checks (see below) at once that $(x, y) \to \langle x|y\rangle$ a positive definite hermitian form (inner product) therefore $\underline{(\mathcal{A}, g)}$ is an Hilbert space. We have $|\langle x|y\rangle| \leq ||x||\,||y||$ and $\varphi(x^*) = \overline{\varphi(x)}$.

4. This inner product satisfies identically $\varphi(x^*(a.y)) = \langle x|a.y\rangle = \langle a^*.x|y\rangle = \varphi((a^*.x)^* y)$ and from that, we get that $\mathcal{A}$ is semi-simple.

Proofs

Proof of 2 and 3.– Linearity on the right is straightforward. To show hermitian symmetry, we first compute $g(x+y,x+y) = g(x,x) + [g(x,y)+g(y,x)] + g(y,y)$ which proves that

$$\Im(g(y,x)) = -\Im(g(x,y)). \tag{10}$$

Then, from,

$$\begin{aligned} &g(x+iy,x+iy) = g(x,x) + [g(x,iy)+g(iy,x)] + g(iy,iy) = \\ &g(x,x) + i[g(x,y)-g(y,x)] + g(y,y) \end{aligned} \tag{11}$$

we get $i[g(x,y)-g(y,x)] \in \mathbb{R}$ meaning $\Re([g(x,y)-g(y,x)]) = 0$. Then $\Re(g(y,x)) = \Re(g(x,y))$ with (10) shows

$$g(y,x) = \overline{g(x,y)} \tag{12}$$

therefore, with $y = e$, we get $\varphi(x^*) = g(x,e) = \overline{g(e,x)} = \overline{\varphi(x)}$. The inequality $|g(x,y)| \leq ||x||.||y||$ is a consequence of Cauchy-Schwartz theorem.

Linear automorphism group on $\mathcal{A}$

• Consider a finite group $H \subset \mathrm{Aut}_{\mathbb{C}}(\mathcal{A})$ of linear automorphisms of the $*$-algebra $\mathcal{A}$ i.e. automorphisms of algebra which commute with the $*$-involution

$$\begin{array}{ll} \forall (h,a) \in H \times \mathcal{A}, & h.(a+b) = h.a + h.b \\ \forall (h,a,b) \in H \times \mathcal{A} \times \mathcal{A}, & h.(ab) = (h.a)(h.b) \\ \forall (h,a,\lambda) \in H \times \mathcal{A} \times \mathbb{C}, & h.(\lambda a) = \lambda h.a \\ \forall (h,a) \in H \times \mathcal{A}, & (h.a)^* = h.a^* \end{array} \tag{13}$$

• φ is called H-invariant if

$$\forall (h,a) \in H \times \mathcal{A}, \qquad \varphi(h.a) = \varphi(a) \tag{14}$$

In other words, φ does not see the action of H.

Linear automorphism group on $\mathcal{A}$

Let φ be a H-invariant SPS on $\mathcal{A}$ a $*$-algebra

1. H is a group of isometries for $g(x,y) = \langle x|y\rangle$:

$$\begin{aligned} \langle h.a|h.b\rangle &= \varphi((h.a)^* h.b) = \varphi((h.a^*)(h.b)) = \varphi(h.(a^*.b)) \\ &= \varphi(a^*.b) = \langle a|b\rangle \end{aligned} \tag{15}$$

2. Consider the elements $H.a := \sum_{h\in H} h.a$. Form the vector space $\kappa(H,\mathcal{A})$ made of orbits (with multiplicities) linearly generated by the vectors $H.a$, for all $a \in \mathcal{A}$:

$$\kappa(H,\mathcal{A}) = \mathrm{Span}_{\mathbb{C}}\{H.a\}_{a\in A} \tag{16}$$

then $\kappa(H,\mathcal{A})$ is a subalgebra of $\mathcal{A}$ which is, moreover $*$-closed.

If $\mathcal{A} = \mathbb{C}(S_n)^{\otimes d}$, $\kappa(H,\mathcal{A}) = \mathcal{K}(n)$ the algebra of ribbon graphs, for a particular φ and H (will prove this in the next section).

$\kappa(H, \mathcal{A})$ is an $*$-algebra

Proof of 2.– We check that the product of vectors stays in $\kappa(H, \mathcal{A})$

$$(H.a)(H.b) = \sum_{h,g \in H} (h.a)(g.b) = \sum_{h,g \in H} h.\Big((a)(h^{-1}g.b)\Big)$$

$$\underset{w=h^{-1}g}{=} \sum_{h \in H} h.\left(\sum_{w \in H} a(w.b)\right) = \sum_{w \in H} H.(a(w.b)) \tag{17}$$

Moreover, $h.a^* = (h.a)^*$ implies $(H.a)^* = H.a^*$, so $\kappa(H, \mathcal{A})$ is $*$-closed.

$\square$

What if $\mathcal{A}$ C^*-algebra ?

Remark.– (i) The reader should be aware that $(\mathcal{A}, g)$ is not necessarily a C*-algebra[1]. In fact, one has the following equivalent conditions:

- $(\mathcal{A}, g)$ is a C*-algebra (i.e. for $||x|| = \sqrt{g(x,x)}$)
- $\dim_{\mathbb{C}}(\mathcal{A}) = 1$

Elementary proof of (i).– A finite dimensional C*-algebra $\mathcal{A}$ is a (finite) direct sum of blocks which are $\mathbb{C}$-algebras of matrices i.e.

$$\mathcal{A} = \oplus_{i=1}^{m} \mathcal{M}(n_i, \mathbb{C}) \tag{18}$$

(see, e.g. [?] Theorem III.1.1). The block, being simple algebras, are therefore two-sided ideals. Hence, decomposing $1_{\mathcal{A}}$ according to (18) yields

$$1_{\mathcal{A}} = \sum_{i=1}^{m} e_i \tag{19}$$

if we had $m > 1$, this would entail that e_1 and $e_2' = \sum_{i=2}^{m} e_i$ be two non-zero orthogonal projectors (orthogonality is proved by means of (9)). Hence, from $||e_1|| = ||e_1 + e_2'|| = ||e_1 - e_2'|| = 1$ we see that $m = 1$ and $n_1 = 1$.

[1]See discussion in `https://math.stackexchange.com/questions/3964927`.

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(ii) However, we can make $\mathcal{A}$ a C*-algebra for the sup norm $||a||_g = \sup_{||\xi||=1} ||a.\xi||$.

• If $\mathcal{A}$ is a C*-algebra then $\kappa(H, \mathcal{A})$ is a C*-algebra.
Proof: This is the consequence of the general fact that an $*$–closed subalgebra of a C*-algebra is a C*-algebra. [Bourbaki Ch 8]

Structure constants, Orbits

- Any basis vector of $\kappa(H, \mathcal{A})$ expands as

$$H.a := \sum_{h \in H} h.a = |\mathrm{Aut}(a)| \sum_{a' \in \mathrm{Orb}(a)} a' \tag{20}$$

where $\mathrm{Aut}(a) = \{h | h.a = a\} \subset H$ is the automorphism subgroup of H that leaves a invariant.

- Orbit-stablizer theorem, we know that $|\mathrm{Aut}(a)| = |H|/|\mathrm{Orb}(a)|$. Also $\forall b \in \mathrm{Orb}(a)$, $\mathrm{Aut}(a) \equiv \mathrm{Aut}(b)$, thus $|\mathrm{Aut}(a)|$ is independent of the representative element in the orbit.

Structure constants. We introduce the following elements:

$$E_a = \frac{1}{|H|} \sum_{h \in H} h.a = \frac{1}{|\mathrm{Orb}(a)|} \sum_{a' \in \mathrm{Orb}(a)} a' \tag{21}$$

and inspect the structure constants

$$E_a E_b = \sum_c C_{ab}^c E_c \,. \tag{22}$$

We want a expansion of C_{ab}^c in terms of orbits of the group action.

Structure constants and central elements

$$\begin{aligned}
E_a E_b &= \frac{1}{|H|^2}\sum_{g\in H}\sum_{h\in H}(g.a)(h.b) = \frac{1}{|H|^2}\sum_{g\in H}\sum_{h\in H} g.(a(g^{-1}h.b)) \\
&= \frac{1}{|H|^2}\sum_{g\in H}\sum_{h\in H} g.(a(h.b)) \qquad\qquad (g^{-1}h\to h) \\
&= \frac{1}{|\mathrm{Orb}(b)|}\frac{1}{|H|}\sum_{b'\in\mathrm{Orb}(b)}\frac{|H|}{|\mathrm{Orb}(a.b')|}\sum_{d\in\mathrm{Orb}(a.b')} d \\
&= \frac{1}{|\mathrm{Orb}(b)|}\sum_{b'\in\mathrm{Orb}(b)}\sum_{c}\frac{1}{|\mathrm{Orb}(a.b')|}\delta(\mathrm{Orb}(c),\mathrm{Orb}(a.b'))\sum_{d\in\mathrm{Orb}(a.b')} d \\
&= \frac{1}{|\mathrm{Orb}(b)|}\sum_{c}\frac{1}{|\mathrm{Orb}(c)|}\cdot\sum_{d\in\mathrm{Orb}(c)} d\sum_{b'\in\mathrm{Orb}(b)}\delta(\mathrm{Orb}(c),\mathrm{Orb}(a.b')) \\
&= \frac{1}{|\mathrm{Orb}(b)|}\sum_{c} E_c\Big(\sum_{b'\in\mathrm{Orb}(b)}\delta(\mathrm{Orb}(c),\mathrm{Orb}(a.b'))\Big) \qquad (23)
\end{aligned}$$

where $\delta(\mathrm{Orb}(p),\mathrm{Orb}(q))$ is the Kronecker delta on orbits. Thus $C^c_{ab} = \frac{1}{|\mathrm{Orb}(b)|}\sum_{b'\in\mathrm{Orb}(b)}\delta(\mathrm{Orb}(c),\mathrm{Orb}(a.b'))$ with

$$\sum_{b'\in\mathrm{Orb}(b)}\delta(\mathrm{Orb}(c),\mathrm{Orb}(a.b')) =$$

Number of times the right multiplication of elements in the orbit b with a fixed element in the orbit a (to the left) produces an element in orbit c.

Structure constants and central elements

There exist particular elements in $\mathcal{A}$ such that

$$T_a = |\mathrm{Orb}(a)| E_a \tag{25}$$

For these elements, we have

$$T_a E_b = |\mathrm{Orb}(a)| \sum_c C_{ab}^c E_c = \sum_c (\mathcal{M}_a)_b^c E_c \tag{26}$$

The following statement is straightforward.

Proposition

Then, for any $a \in \kappa(H, \mathcal{A})$, the matrix elements $(\mathcal{M}_a)_b^c$ are non negative integers.

Question:
Are there some T_a that generate the center of $\kappa(H, \mathcal{A})$?

$\mathcal{A}$ a group algebra

In the case that $\mathcal{A} = \mathbb{C}(\mathcal{G})$,

→ Consider $\mathcal{G}$ a finite group, and $\mathcal{A} = \mathbb{C}(\mathcal{G})$ its group algebra.
→ Case of interest $\mathcal{A} = \mathbb{C}(G)^{\otimes d} \simeq \mathbb{C}(G^{\times d})$ is an algebra. We write for simplicity $\mathcal{G} = G^{\times d}$.

• $*$-involution on $\mathcal{A}$

$$X^* = \sum_{g\in\mathcal{G}} \bar{a}_g g^{-1} \,, \tag{27}$$

• PSP φ: Given $X = \sum_{g\in\mathcal{G}} a_g g \in \mathcal{A}$, $\varphi : \mathcal{A} \to \mathbb{C}$

$$\varphi(X) := a_e \,, \quad \text{pick the coeff of the identity}$$

$$X^*X \in \mathcal{C}(\mathcal{A}), \qquad \varphi(X^*X) = \sum_{g\in\mathcal{G}}\sum_{h\in\mathcal{G}} \bar{a}_g a_h \varphi(g^{-1}h) \tag{28}$$

pick all coefficients such that $[e = g^{-1}h] \;\Rightarrow\; (g = h)$. Thus $\varphi(X^*X) = \sum_{g=h} |a_h|^2 \geq 0$.

• $(\mathcal{A}, *)$ is a $*$-algebra that is semi-simple. Semi-simpliciy feature is mainly the Maschke theorem (let G be a finite group and k a field whose characteristic does not divide the order of G. Then $k[G]$, the group algebra of G, is semisimple).

$\mathcal{A}$ a group algebra

• δ is the Kronecker delta function on $\mathcal{G}$ ($\delta(g) = 1$ if and only if $g = e$, otherwise $\delta(g) = 0$).
• A sesquilinear form on $\mathcal{A}$ as

$$\Big\langle \sum_{g\in\mathcal{G}} a_g g, \sum_{h\in\mathcal{G}} a_h h \Big\rangle = \sum_{g,h\in\mathcal{G}} \bar{a}_g a_h \, \delta(g^{-1}h) \qquad (29)$$

Proposition

For any $X, Y \in \mathcal{A}$,

$$\langle X, Y\rangle = \varphi(X^* Y) \qquad (30)$$

Proof : $\varphi(g) = \delta(g)$, for all $g \in \mathcal{G}$.

Linear automorphism group of $\mathcal{A}$

- H the subgroup of $\mathcal{G}$, defined by the adjoint action: $\forall(h,g) \in H \times \mathcal{G}$

$$g \mapsto \quad hgh^{-1} \tag{31}$$

- The action of H on $\mathcal{A}$ extends by linearity on $\mathcal{A}$

$$h.X = \sum_{g \in \mathcal{G}} a_g \, hgh^{-1} \, . \tag{32}$$

- H commutes with the $*$-involution: $(h.X)^* = hX^*$.

Proposition

$\forall(h,X) \in H \times \mathcal{A}, \qquad \varphi(h.X) = \varphi(X).$

Proof: Consider a couple $(h,X) \in H \times \mathcal{A}$

$$\varphi(h.X) = \sum_{g \in \mathcal{G}} a_g \, \varphi(hgh^{-1}) = \sum_{g \in \mathcal{G} \,|\, hgh^{-1} = e} a_g = a_e = \varphi(X) \tag{33}$$

- H is an isometry group of $\mathcal{A}$.
- $\kappa(H,\mathcal{A})$ is a $*$-sublagebra of $\mathcal{A}$.
- The restriction $\varphi|_{\kappa(H,\mathcal{A})}$ is a SPS for $\kappa(H,\mathcal{A})$ and thereby proves that $\kappa(H,\mathcal{A})$ is semi-simple.
- The inner product of $\mathcal{A}$ should restrict on $\kappa(H,\mathcal{A})$

T_a operators

- Special base elements

$$T_a = \sum_{g \in \mathcal{C}_a} g \tag{34}$$

where $\mathcal{C}_a$ is a particular conjugacy class. The label 'a' here is yet to be determine.

- A sufficient number of these elements generates the center of $\kappa(H, \mathcal{A})$.

- In the case: $G = S_n$:

$\to \mathcal{C}_a =$ conjugacy class with 1 cycle of size a and all remaining cycles of size 1.
$n = 3$, $\quad \mathcal{C}_2 = \{(12)(3);\ (13)(2);\ (23)(1)\}$.

$\to$ T_a's commute with each other

$$T_a T_b = T_b T_a \tag{35}$$

for a few number of T_a's, $a = 2, 3, \ldots, n-1$, $\{T_a\}$ generates the center of $\mathbb{C}(S_n)$.

Outline

$\mathcal{K}^\infty$ algebra

- An infinite dimensional associative algebra obtained by summing $\mathcal{K}(n)$ over n

$$\mathcal{K}^\infty = \bigoplus_{n=0}^{\infty} \mathcal{K}(n) \tag{36}$$

- Two associative products on this vector space.
- The product at fixed n: $\mathcal{K}^\infty$ is an associative semi-simple algebra?
- Outer product on $\mathcal{K}^\infty$:

$$\begin{aligned}
&E_{(\overline{\sigma}_1,\overline{\sigma}_2)} = \sum_{\gamma_1 \in S_{n_1}} \gamma_1 \sigma_1 \gamma_1^{-1} \otimes \gamma_1 \sigma_2 \gamma_1^{-1} \in \mathcal{K}(n_1) \\
&E_{(\overline{\tau}_1,\overline{\tau}_2)} = \sum_{\gamma_2 \in S_{n_2}} \gamma_2 \tau_1 \gamma_2^{-1} \otimes \gamma_2 \tau_2 \gamma_2^{-1} \in \mathcal{K}(n_2) \\
&\circ : \mathcal{K}(n_1) \otimes \mathcal{K}(n_2) \to \mathcal{K}(n_1 + n_2) \\
&E_{(\overline{\sigma}_1,\overline{\sigma}_2)} \circ E_{(\overline{\tau}_1,\overline{\tau}_2)} = \sum_{\gamma \in S_{n_1+n_2}} \gamma(\overline{\sigma}_1 \circ \overline{\tau}_1)\gamma^{-1} \otimes \gamma(\overline{\sigma}_2 \circ \overline{\tau}_2)\gamma^{-1} = E_{(\overline{\sigma}_1 \circ \overline{\tau}_1),(\overline{\sigma}_2 \circ \overline{\tau}_2)}
\end{aligned} \tag{37}$$

This outer product is related to the ring structure which has been described in detail, using the representation basis in [de Mello Koch et al, arXiv:1707.01455 [hep-th]].

- More products, co-product and Hopf algebra structure?